ON THE DISPLACEMENTS OF A DEFORMABLE BODY IN AN ACOUSTIC MEDIUM

(O PEREMESHCHENII DEFORMIRUEMOGO TELA V AKUSTICHESKOI SREDE)

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The problem of the displacements of a deformable body under the action of unsteady acoustic waves of arbitrary form is solved herein.

Some general conclusions are drawn regarding the effect of the properties of the medium and of the deformations of the body on its final displacement (at $t = \infty$).

In particular, it is shown that the final displacements of a body immersed in a viscous fluid do not depend on its buoyancy.

Novozhilov [1] investigated the problem of the displacements of a body of arbitrary shape immersed in a fluid and subjected to acoustic pressure waves.

That problem was solved under the following assumptions: an absolutely rigid body, an ideal fluid, and a plane pressure wave.

The effects on the displacement of the body of its deformability and the (time-independent) properties of the acoustic medium (for example, the viscosity of the fluid) are clarified below. In general, there are no assumptions as to the form of the wave.

The linearity of the problem allows the equations of motion of the body to be written in terms of components along its principal central axes of inertia in the following manner:

$$M_{ii}\ddot{u}_{*i} + Q_i = P_i \qquad (i = 1, \ldots, 6) \tag{1}$$

Here M_{ii} is the mass (moment of inertia) of the body, $\ddot{u}_{*i}$ is the acceleration of the center of mass of the body (the ratio of the moment of the inertia force to the moment of inertia is the angular acceleration of the deformable body), P_i is the force (moment) exerted on an absolutely rigid immovable body by the waves propagating in the medium, and Q_i is the additional force (moment) of interaction of the body with the medium caused by the displacement of the body (more precisely, by the displacement and deformation of the surface of the body); dots denote differentiation with respect to time t.

The force (moment) Q_i is determined by the integral over the surface S of the body of the scalar product of the pressure $\mathbf{q}$ causing displacement of the body with the unit vector $\mathbf{v}_i$ directed along the corresponding axis (the vector function of the displacement of a point on the surface of the undeformed body for a unit rotation about the corresponding axis)

$$Q_i = \iint\limits_{(S)} \mathbf{q}\,(x, y, t) \cdot \mathbf{v}_i(x, y)\, ds \tag{2}$$

where x and y are the coordinates on the surface and t is the time.

The dependence of the generalized force Q_i on the displacement of the surface may be expressed in an explicit form. In order to do this it is convenient to represent the displacement of the surface in the following form:

$$\mathbf{w}\,(x, y, t) = \sum_k u_k\,(t)\,\mathbf{v}_k\,(x, y) \qquad (k = 1, 2, \ldots) \tag{3}$$

Here $\{\mathbf{v}_k(x, y)\}$ is a sufficiently complete set of vector functions such that for $k = 1, \ldots, 6$ these functions coincide with the functions $\mathbf{v}_i$ determined above; that is, they correspond to the displacements of the surface as a whole, while the remainder ($k = 7, 8, \ldots$) represent the deformation of the surface; $u_k(t)$ is a generalized coordinate. In the absence of deformation $u_k = u_{*k}$ for $k = 1, 2, \ldots, 6$, $u_k = 0$ for $k = 7, 8, \ldots$.

Let the surface of the body displace or deform such that the generalized coordinate u_k increases with unit speed

$$\dot{u}_k = 1 \quad \text{for } t > 0, \qquad u_k = 0 \qquad \text{for } t < 0$$

$u_m = 0 \quad$ for $m \neq k$.

In general, this process gives rise to a pressure on the surface of the body with components along each of the directions $\mathbf{v}_i$. Relation (2) defines the generalized force $F_{ik}(t)$ corresponding to these conditions.

Owing to the linearity of the problem, the generalized force $Q_i(t)$ arising in an arbitrary displacement of the surface is determined by an equality which is an immediate consequence of the superposition principle

$$Q_i(t) = \sum_k Q_{ik}(t) = \sum_k \int_0^t F_{ik}(t-\tau)\, \ddot{u}_k(\tau)\, d\tau \qquad (4)$$

Here and henceforth it is assumed that $u_k = 0$ for $t \leqslant 0$, while $\ddot{u}_k$, generally speaking, includes impulsive functions (in particular, if $\lim u_k \neq 0$ as $t \to +0$).

The dependence of the generalized force P_i on the parameters of the wave may also be exhibited with the help of the functions F_{ik}.

An examination of the portion of the medium bounded (conceptually) by the surface of the body enables the determination of this dependence.

The dynamic equilibrium of the "fictitious body" obtained in this manner may be described by equations similar to equations (1).

If the body under consideration is replaced by the fictitious body and its equations of motion are referred to the axes chosen above, then the components of the external forces on the undisplaced surface and the functions F_{ik} for the fictitious body will be the same as those for the actual body,* and these equations have the form

$$\sum_n M_{ni}^{\circ}\, \ddot{u}_{*n}^{\circ} + \sum_k \int_0^t F_{ik}(t-\tau)\, \ddot{u}_k^{\circ}(\tau)\, d\tau = P_i \qquad (5)$$

Here M_{ni}° is the mass, statical moment, or moment of inertia of the fictitious body about said axes. The superscript $^{\circ}$ denotes quantities associated with the fictitious body.

The moment of the inertia force is written here as the sum of the moments due to the generalized displacements u_{*n}°, for both $n = i$ and $n \neq i$, because the axes relative to which the equations of motion of the fictitious body are written are in general not its principal central axes of inertia.

The fictitious body does not cause any disturbances in the wave propagation through the medium. Hence the necessary data regarding its

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* It is assumed that the body does not "separate" from the medium. If the body is in a solid elastic medium or a real fluid, the displacements of the medium on the surface of the body are identical to the displacements of the body. In the case of an ideal fluid this condition applies to the normal displacements.

displacements (u_k°, u_{*i}°) may be obtained by an appropriate integration of the projected displacements of the medium over the surface and volume of the body.

Thus equations (5) may be considered as equalities determining the force P_i.

On the basis of relations (4) and (5) the equations of motion of the body (1) may be written in the following form:

$$M_{ii}\,\ddot{u}_{*i} + \sum_k \int_0^t F_{ik}(t-\tau)\,\ddot{u}_k(\tau)\,d\tau = \sum_n M_{ni}^{\circ}\,\ddot{u}_{*n}^{\circ} + \sum_k \int_0^t F_{ik}(t-\tau)\,\ddot{u}_k^{\circ}(\tau)\,d\tau \qquad (6)$$

An analysis of equations (6) makes possible certain general conclusions regarding the final displacements of the body.

The Laplace transform reduces equations (6) to the form

$$M_{ii}\,p^2 u_{*i}^{+} + \sum_k F_{ik}^{+}\,p^2\,u_k^{+} = \sum_n M_{ni}^{\circ}\,p^2 u_{*n}^{\circ +} + \sum_k F_{ik}^{+} p^2 u_k^{\circ +} \qquad (7)$$

where the meanings of the superscript $^{+}$ and the parameter p are defined by the relation

$$\int_0^{\infty} \Phi(t)\,e^{-pt}\,dt = \Phi^{+}(p)$$

In order to determine the magnitudes of the final displacements of the body from equations (7), it is sufficient to use formula [2]

$$\lim_{t\to\infty} \Phi(t) = \Phi_{\infty} = \lim_{p\to 0} p\Phi^{+}(p) \qquad (8)$$

which is valid provided the limit (which need not be bounded) on the left-hand side exists and satisfies the condition

$$\lim_{t\to\infty} \Phi(t)\,e^{-pt} = 0 \quad \text{for } p > 0$$

If it is assumed that the wave due to the external disturbance is bounded in time or is damped, while the medium is unbounded, one may assert that the displacements of the body will satisfy the above conditions.

Actually, the disturbance waves emitted due to the vibrations of a body in an unbounded medium cause these vibrations to be damped. Hence, if vibrations of the body occur, then they should damp out when the external disturbance stops, and consequently the displacements of the body will approach a limit. Moreover, the boundedness of the external influences precludes the exponential growth of the displacements of the body as $t \to \infty$.

The character of the relation between the final displacements of the body and the final displacement of the fictitious body (or the displacements of the medium in the absence of a body), as follows from relations (7) and (8), depends to a considerable extent on the behavior of the function $F_{ik}^{+}(p)$ as $p \to 0$ ($F_{ik}(t)$ as $t \to \infty$).

In the case of a body of finite dimensions, the functions F_{ik} may be classified as follows, depending on the properties of the (unbounded) medium.

1. Ideal fluid. For $t > 0$ let the body move (deform) with unit velocity $\dot{u}_k = 1$ in an infinite ideal fluid ($u_k = 0$ for $t < 0$). After a sufficiently long time, when the flow has been established, the compressibility of the fluid will not yet have affected the velocity field in a sufficiently large neighborhood of the body, and the momentum of the fluid will be characterized by the additional masses m_{ik}. The motion of the liquid is caused by the forces F_{ik} with $\dot{u}_k = 1$, hence we have

$$\lim_{t\to\infty} \int_0^t F_{ik}(\tau)\, d\tau = m_{ik} \tag{9}$$

Thus, in the case of an ideal fluid the functions F_{ik} are integrable and application of formula (8) gives

$$\lim_{p\to 0} F_{ik}^{+}(p) = \int_0^\infty F_{ik}(\tau)\, d\tau = m_{ik} \tag{10}$$

2. Real fluid. Uniform motion in a real fluid will be opposed by friction forces α_{ik}, hence we have

$$\lim_{t\to\infty} F_{ik}(t) = \alpha_{ik} \tag{11}$$

One may assert that $\alpha_{ii} > 0$. Moreover, it is clear that the friction forces α_{ik} are bounded.

In accordance with formula (8)

$$\lim_{p\to 0} pF_{ik}^{+} = \alpha_{ik} \tag{12}$$

3. Solid elastic medium. From the relations

$$Q_{ik}^{+} = pF_{ik}^{+}\, pu_k^{+} = (\dot{F}_{ik})^{+} (\dot{u}_k)^{+} \tag{13}$$

it is clear that $\dot{F}_{ik}$ is the generalized force corresponding to a unit displacement $u_k = 1$ for $t > 1$ ($u_k = 0$ for $t < 0$).

For a solid elastic medium the limit of this force β_{ik} (as $t \to \infty$) differs from zero at least for $k = i$. In this case $\beta_{ii} > 0$. Moreover, it is clear that all the coefficients of rigidity β_{ik} are bounded.

Application of formula (8) to $\dot{F}_{ik}$ gives the following relation:

$$\lim_{p\to 0} p^2 F_{ik}^{+} = \beta_{ik} \tag{14}$$

Now it is possible to consider the limiting relations (as $p \to 0$) which follow from system (7) and determine the final displacements of the body.

If all terms of equation (7) are multiplied by p^q, where $q = -1, 0, 1$ according to the cases considered above, and p is made to approach zero, the limiting relations will have the form

a) $$M_{ii}\,u_{i\infty} + \sum_{k=1}^{6} m_{ik}\,u_{k\infty} + [M_{ii}\,(u_{*i\infty} - u_{i\infty})] + \Big[\sum_{k=7}^{\infty} m_{ik}\,u_{k\infty}\Big] = \tag{15}$$

$$= \sum_{n} M_{ni}^{\circ}\,u_{n\infty}^{\circ} + \sum_{k=1}^{6} m_{ik}\,u_{k\infty}^{\circ} + \Big[\sum_{n} M_{ni}^{\circ}\,(u_{*n\infty}^{\circ} - u_{n\infty}^{\circ})\Big] + \Big[\sum_{k=7}^{\infty} m_{ik}\,u_{k\infty}^{\circ}\Big]$$

b) $$\sum_{k=1}^{6} \alpha_{ik}\,u_{k\infty} + \Big[\sum_{k=7}^{\infty} \alpha_{ik}\,u_{k\infty}\Big] = \sum_{k=1}^{6} \alpha_{ik}\,u_{k\infty}^{\circ} + \Big[\sum_{k=7}^{\infty} \alpha_{ik}\,u_{k\infty}^{\circ}\Big] \tag{16}$$

c) $$\sum_{k=1}^{6} \beta_{ik}\,u_{k\infty} + \Big[\sum_{k=7}^{\infty} \beta_{ik}\,u_{k\infty}\Big] = \sum_{k=1}^{6} \beta_{ik}\,u_{k\infty}^{\circ} + \Big[\sum_{k=7}^{\infty} \beta_{ik}\,u_{k\infty}^{\circ}\Big] \tag{17}$$

The terms in the square brackets correspond to the residual deformations of the body (on the left-hand side of the equation) and of the fictitious body (on the right).

Examination of these relations leads to the following conclusions:

1) Elastic deformations do not affect the final displacements.

Actually, all of the coefficients M_{ii}, m_{ik}, α_{ik} and β_{ik} are bounded, while the residual deformations vanish, hence the terms corresponding to them also vanish.

2) In the case where the functions F_{ii} are not integrable (a real fluid or a solid elastic medium), the mass of the body and the deformations within the body have no effect on the final displacements of its surface.

This result is a consequence of equations (16) and (17), in which the generalized masses of the body do not appear.

3) If the deformations of both bodies are elastic, and their corresponding principal central axes and masses (moments of inertia) are the same, the final displacements of the body are equal to the final displacements of the fictitious body (or medium without a body).

The deformations of the fictitious body will be elastic, for example, in the case of a plane wave, where all particles in the medium are displaced by the same amount.

4) As follows from (16) and (17), the final displacements of the surface of the body in a real fluid or in an elastic solid are the same as the final displacements of the surface of the fictitious body, provided that the deformations of the surfaces of both bodies (u_k and u_k° for $k = 7, 8, \ldots$) are elastic.

We now turn our attention to the principal difference of systems (16) and (17) from system (15). Terms containing the generalized masses of the body are absent from the former (a fact which enabled Conclusions 2 and 4 to be drawn). This difference depends on the physical nature of the equations.

Equations (17) are equations for the generalized force (at $t = \infty$). The right-hand sides of the equations represent forces acting on the undisplaced body owing to the displacements of the medium, while the left-hand sides are necessary for the displacements of the body in an undisturbed medium. The equality of these forces is evident if one considers that the first state may be obtained by a displacement of the body from the final to the initial position.

It is also clear that the mass of the body will have no effect whatsoever on those forces arising from the elasticity of the medium.

The similar situation in the case of a real fluid is less clear. Equations (16) are equalities between integrals of the generalized forces with respect to time - they are equalities between the final values of the impulses

$$\int_0^\infty Q_{ik}(t)\,dt$$

In the case of a real fluid the impulses necessary for the displacement of the body (the left-hand sides of the equations) or for the prevention of it (right-hand side) are finite and non-zero. The impulse of the inertia force of the body necessary for its displacement from one state of rest into another is equal to zero. Hence, the generalized masses of the body do not have an effect on its final displacement and do not appear in system (16). The same is true of the additional mass.

In this sense a real fluid is closer to an elastic solid than to an ideal fluid.

In the case of an ideal fluid the final impulse equals zero. Equations (15) are equalities of the integrals of the impulses

$$\int_0^\infty \int_0^t Q_{ik}(\tau)\, d\tau dt$$

For the inertia forces of the body and the fluid (the added mass), these quantities are finite and non-zero. As a result, the generalized masses of the body and the added mass appear in system (15) and influence the final displacements of the body.

Let an elastic body possess such symmetry that under the action of a plane wave in an ideal fluid all of its generalized displacements except one are zero. In this case, from equation (15), follows the formula of Novozhilov [1]

$$u_\infty = \frac{M^\circ + m}{M + m} u_\infty{}^\circ \tag{18}$$

which is seen to be valid for an elastic body as well.

From formula (18) it is clear that the final displacement of the elastic body under the action of a plane wave in an ideal fluid depends on its buoyancy.

For positive buoyancy ($M < M^\circ$), it is greater than the displacement of a particle of the fluid, while for negative buoyancy ($M > M^\circ$) it is less.

On the other hand, as shown above, the displacement of a body under the same conditions but in a real fluid does not depend on its mass and always equals the displacement of a particle of the fluid.

In this connection one may raise a question regarding the applicability of formula (18) for real conditions. The answer to this question is the following.

In an ideal fluid let the body at time T attain a displacement sufficiently close to the final displacement, so that subsequently it does not depart from a negligibly small neighborhood of the final state.

If the viscosity of a real fluid is sufficiently small or the dimensions of the body are sufficiently large that the inertia forces arising in the displacement of the body during the time $0 < t < T$ predominate over the friction forces, then formula (18) is applicable. It gives the value of the final displacement in that interval during which the inertia forces predominate. However, as $t \to \infty$ the displacement of the body in a real fluid approaches the displacement of a fluid particle independently of its buoyancy. Hence, during the remaining time ($t > T$), a body of positive buoyancy which is displaced more than the fluid, returns (although slowly for small friction) so that its displacement becomes equal to the displacement of a fluid particle.

These statements may be illustrated by the following example. Let an absolutely rigid body of mass M be displaced in a real incompressible (for simplicity) fluid in one direction, where the functions defining the displacement of the medium and the interaction of the body with the medium have the forms

$$F = m\delta_1(t) + \alpha, \qquad \dot{u}_*^\circ = \dot{u}^\circ = \delta_0(t) - \delta_0(t-1)$$

$$\delta_0(z) = 0 \quad \text{for } z < 0, \qquad \delta_0(z) = 1 \quad \text{for } z > 0$$

$$\dot{\delta}_0(z) = \delta_1(z), \qquad \int_0^a \delta_1(z)\,dz = 1 \quad \text{for } a > 0, \qquad \delta_1(z) = 0 \quad \text{for } z \neq 0$$

Here m is the added mass and α is the coefficient of friction. The final displacement of a fluid particle is

$$u_\infty^\circ = 1 \ \ (u^\circ(t) = u_\infty^\circ = 1 \ \text{for } t \geqslant 1)$$

The given information enables one to obtain the following equation of motion of the body from system (6):

$$(M + m)\ddot{u} + \alpha\dot{u} = (M^\circ + m)[\delta_1(t) - \delta_1(t-1)] + \alpha[\delta_0(t) - \delta_0(t-1)] \qquad (19)$$

The solution of this equation is (for $t > 0$)

$$u = \frac{M^\circ + m}{\alpha}[1 - e^{-\nu t} - (1 - e^{-\nu(t-1)})\delta_0(t-1)] + t - \frac{1}{\nu}(1 - e^{-\nu t}) - \left[t - 1 - \frac{1}{\nu}(1 - e^{-\nu(t-1)})\right]\delta_0(t-1) \qquad \left(\nu = \frac{\alpha}{M + m}\right) \qquad (20)$$

This dependence may be represented by the series (for $t \geqslant 1$)

$$u = \frac{M^\circ + m}{M + m} + \frac{M - M^\circ}{M + m}\sum_{n=1}^{\infty}(-1)^{n+1}\nu^n \frac{t^{n+1} - (t-1)^{n+1}}{(n+1)!} \qquad (21)$$

It is clear that for $\nu T \ll 1$, where T may be large for sufficiently small friction, the displacement of the body during the time interval $1 \leqslant t \leqslant T$ is determined by the first term of expression (21), that is, formula (18).

However, for subsequent increases in time t, as is clear from equation (21), the displacement u of the body decreases if $M < M^0$ and increases if $M > M^0$, and, as follows from the expression (20)

$$\lim u = u_\infty = 1 = \overset{\circ}{u}_\infty \quad \text{for } t \to \infty$$

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